Network Reliability

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Part 2: Mathematical Modelling Scenarios

DEFINING RELIABILITY: DESIGNING RELIABLE NETWORKS
Presentation Outline

• Definitions of reliability
• Some reliability models
  – Deterministic models
  – Probabilistic models
Reliability is concerned with the ability of a network to carry out a desired operation such as "communication".

- Reliability measures can be categorised as:
  - A source node communicating with a terminal node for all OD pairs. **Terminal reliability**.
  - k (>1) nodes communicating with a terminal node. k-Terminal or **Many source to terminal reliability**.
  - Source node communicating to k operating terminal nodes. **Source to many terminal reliability**.
  - All operative nodes communicating. **All terminal or Network Reliability**.
Definitions of Reliability - 2

- Additional measures of reliability include:
  - Node to node blocking
  - Node to node delay
  - Average terminal reliability
  - Functional reliability
  - ...
Reliability Models

• Models involving reliability can be classified into two main types:

• **Deterministic Models**
  – In this class we include:
    • Connectivity
    • Hop counts
    • Node and link disjoint paths

• **Probabilistic Models**
  – In this class we include:
    • Path reliabilities
    • Probability of path disconnection
Introduction

• **The Problem:**
  – How can we improve the reliability of our communication networks?

• **Solutions:**
  – Improve the reliability of the individual components of the network.
  – Increase the number of alternative paths available to origin-destination (OD) pairs.
  – Decrease the number of intermediate links between the origin and the destination nodes.

• **Constraints:**
  – Cost
  – Performance
Deterministic Models - Connectivity

- A network is *connected* if there is a path from every node to every other node in the network.
- Intuitively, if all nodes were connected together with a piece of string, then a connected graph/network would stay in *one* piece when we picked it up!
Some terminology

- A **simple path** is a path in which no node is repeated.
- A **cycle** is a path that is simple except that the first and last node are the same.
- A graph with no cycles is called a **tree**.
- A group of disconnected trees is called a **forest**.
- A **spanning tree** of a graph is a subgraph that contains all the nodes, but only enough links to form a tree.
Biconnected Networks

• **Definition:**
  – A network is said to be *biconnected* if for every node in the network, there are at least two distinct paths connecting them.

• An *articulation point* in a connected graph is a node that, if deleted, would break the graph into two or more pieces.

• A graph with no articulation points is said to be *biconnected*.
Example of Biconnected Graph

The articulation points are: \{A, H, J, G\}

Can you find the articulation points?

The articulation points are: \{A, H, J, G\}
Finding Articulation Points with Software

Example of a network that is not biconnected

Notice that there are several network connectivity failure points on this network and their labels are highlighted in red boxes.

These points are at Routers A, H, J and G
Node Disjoint Paths

• The figure shows a node disjoint path
• Except for the start and end nodes, there are no nodes in common in the two paths.
Link Disjoint Paths

- The figure shows two link disjoint paths (red and green)
- There are no links in the common path

Note that there is a common node in the two paths!
To find link disjoint paths in a network we perform the following steps:

- Assign *unit* capacities to each link in the network
- Perform a *maximum flow algorithm* for the modified network

Note that the paths may *not* be unique.
Transformation from Node Disjoint to Link Disjoint Problem

- We replace a node with two nodes joined by a link with *unit capacity*
- This will ensure that only *one path* passes through the original node.
Restricting the Number of Paths

- Through a transmission group
  - Add a new node and link.
  - The link is given capacity R
  - The transmission group members are assigned unit capacity

- Through a node
  - Add a new node and link.
  - The link is given capacity R

Network Reliability Slide 17
Survivability Model - Maximum Flow

- How can we determine the most vulnerable parts of our network?
- How much traffic will fail to reach its destination in the event of a link or node failure?
- We can answer these questions by using an approach involving the maximum flow algorithm.
Probabilistic Models - Path Reliability

• Suppose that we are interested in finding the most reliable path through a network.
• Imagine that we have the individual failure probabilities (or their inverse) for each link and node of the network.
• How can we use this information to locate the most reliable path?

WARNING: Mathematics ahead!
Simple Example

- Let $q =$ probability of link failure
- Define $p = 1 - q$
- Assume perfectly reliable nodes:

$$P_l = p^l$$

The probability that a single path consisting of $l$ links is available is given by $P_l = p^l$. 
Influence of Path Length and Probability of Link Failure

<table>
<thead>
<tr>
<th>Length</th>
<th>P{Failure} 0.001</th>
<th>P{Failure} 0.002</th>
<th>P{Failure} 0.005</th>
<th>P{Failure} 0.01</th>
<th>P{Failure} 0.02</th>
<th>P{Failure} 0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9990</td>
<td>0.9980</td>
<td>0.9950</td>
<td>0.9900</td>
<td>0.9800</td>
<td>0.9500</td>
</tr>
<tr>
<td>2</td>
<td>0.9980</td>
<td>0.9960</td>
<td>0.9900</td>
<td>0.9801</td>
<td>0.9604</td>
<td>0.9025</td>
</tr>
<tr>
<td>3</td>
<td>0.9970</td>
<td>0.9940</td>
<td>0.9851</td>
<td>0.9703</td>
<td>0.9412</td>
<td>0.8574</td>
</tr>
<tr>
<td>4</td>
<td>0.9960</td>
<td>0.9920</td>
<td>0.9801</td>
<td>0.9606</td>
<td>0.9224</td>
<td>0.8145</td>
</tr>
<tr>
<td>5</td>
<td>0.9950</td>
<td>0.9900</td>
<td>0.9752</td>
<td>0.9510</td>
<td>0.9039</td>
<td>0.7738</td>
</tr>
<tr>
<td>6</td>
<td>0.9940</td>
<td>0.9881</td>
<td>0.9704</td>
<td>0.9415</td>
<td>0.8858</td>
<td>0.7351</td>
</tr>
<tr>
<td>7</td>
<td>0.9930</td>
<td>0.9861</td>
<td>0.9655</td>
<td>0.9321</td>
<td>0.8681</td>
<td>0.6983</td>
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<tr>
<td>8</td>
<td>0.9920</td>
<td>0.9841</td>
<td>0.9607</td>
<td>0.9227</td>
<td>0.8508</td>
<td>0.6634</td>
</tr>
<tr>
<td>9</td>
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<td>0.9821</td>
<td>0.9559</td>
<td>0.9135</td>
<td>0.8337</td>
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</tr>
<tr>
<td>10</td>
<td>0.9900</td>
<td>0.9802</td>
<td>0.9511</td>
<td>0.9044</td>
<td>0.8171</td>
<td>0.5987</td>
</tr>
</tbody>
</table>

The above table shows the impact of length upon the reliability of a connection in a series list of nodes and links.
Find a Path with Maximum Reliability - 1

• Suppose that we have a set of links $R_i$ representing the paths between an OD pair.
• Let $p_k$ be the probability that link $k$ is available.

• The highest reliability path satisfies:

$$P_{\ell} = P\{\text{Link available}\} = \prod_{k \in R_{\ell}} p_k$$

• This is equivalent to saying

$$P_{\ell} = \max_k \{ P_k \}$$

$$P_{\ell} = \min_{\ell} \left\{ \frac{1}{P_{\ell}} \right\}$$
Find a Path with Maximum Reliability - 2

- Taking logarithms and noting that the log of a probability will be *negative*, we see that this problem is transformed into a shortest path model:

- With link costs given by:

\[

c_{ij} = (-\log p_j)
\]

\[
\min_{\ell} \left\{ -\sum_{j \in R_{ij}} \log p_j \right\}
\]
Reliable Private Network Design Model

• We consider a generic model for determining a "reliable network design" based on the requirement that each OD pair must have a primary and a secondary path.

• Applications for this approach include:
  – Networks dimensioned on a demand basis.
  – Private circuit switched networks.
  – Private packet switched networks.
  – Cellular mobile networks employing "directed retry".
  – Common Channel Signalling networks.
  – B-ISDN networks for the selection of primary and stand-by virtual paths.
Generic Model - Outline

• Reliability:
  – For each OD pair in the network, we shall nominate a primary path for the demand to use and a secondary (backup) path to be used in the event of a single link failure.
  – The primary and secondary paths are to be link disjoint.
  – For each link we shall assign a probability of being operational, \((1 - \beta)\).
  – In the event that two paths have equal costs, we shall select the path with the highest reliability.

• Performance:
  – The user nominates the performance standards required for both the primary and secondary paths.
Generic Model - Notation

- L: The index set of candidate links
- R: The set of candidate paths for the model
- \( \Pi \): The set of OD pairs for the network
- \( S_p \): The set of candidate paths for OD pair p
- \( n_l \): The capacity of the installed link l
- \( F_l \): The flow on link l
- \( C(n_l) \): The cost of link l with respect to the installed capacity
- \( \beta_l \): The probability that the link is down
- \( x_r^p \): The binary decision variable which is 1 if path r is chosen for OD pair p as the primary path.
- \( u_r^p \): The binary decision variable which is 1 if path r is chosen for OD pair p as the secondary path.
Generic Model Specification - 1

• **Objective Function:**

\[ \sum_{l \in L} C(n_l) \]

The exact nature of this objective function is left unspecified in the generic model and suitable functions are selected for appropriate applications.

• **Constraints:**

\[ \sum_{l \in L} x_r u_q \delta_{rl} \delta_{ql} = 0 \quad \forall r, q \in S_p, p \in \Pi \]

This constraint specifies that the links for the primary and secondary paths must be disjoint.
Generic Model Specification - 2

Where

- $\Delta_{li}$ : The increase in flow on link $l$ as a result of failure on link $i$
- $\xi_{li}$ : The decrease in flow on link $l$ as a result of failure on link $i$.

\[
F_i + \Delta_{li} - \xi_{li} \leq n_i \quad \forall l \neq i \in L
\]

This constraint says that the capacity of link $l$ is not exceeded during single link failures of link $i$. 
Generic Model Specification - 3

- The following formulae are used for computing the increases and decreases in link flows for link l under single link failure of link i.

\[ \Delta_{li} = \sum_{p \in \Pi} \sum_{r \in S_p} \gamma_p (1 - G_p) x_r^p u_q^p \delta_{ri}^p \delta_{ql}^p \quad \forall l \neq i \in L \]

and

\[ \xi_{li} = \sum_{p \in \Pi} \sum_{r \in S_p} \gamma_p (1 - G_p) x_r^p \delta_{ri}^p \delta_{rl}^p \quad \forall l \neq i \in L \]
Under normal operating conditions,

\[ F_i \leq n_i \quad \forall l \in L \]

This constraint simply says that the capacity of link \( l \) must be sufficient under normal load conditions.

The flow on link \( l \) is determined from:

\[ F_l = \sum_{p \in \Pi} \sum_{r \in S_p} x_r^p \delta_{r l}^p \gamma_p (1 - G_p) \]

Where

- \( \gamma_p \) is the offered traffic demand for OD pair \( p \)
- \( G_p \) is the end to end grade of service under normal conditions.
• We also require that

\[ \sum_{r \in S_p} x_r^p = 1, \quad \forall p \in \Pi \]

Which ensures that one primary path only per OD pair is selected by the model.

\[ \sum_{r \in S_p} u_q^p = 1, \quad \forall p \in \Pi \]

Which ensures that one secondary path only per OD pair is selected by the model.

\[ x_r^p, u_q^p \in \{0,1\} \]

\[ n_l \geq 0 \quad \text{and integer, } l \in L \]
Linear Model

- The simplest form of application is the linear model in which the objective function may be written as:

\[ \sum_{l \in L} c_i \frac{n_i}{(1 - \beta_i)} \]

- In this case, the model is quite easy to solve and a simple algorithm has been devised to implement the solution procedure.
- Such a model could be useful where circuit demands are specified in absolute terms such as total calls or kbits/sec.
- Notice that we can take the reliability of the links into account by incorporating the special term in the denominator.
Circuit Switched Network Model

\[ \sum_{l \in L} c_l \frac{n_l(f_l)}{(1 - \beta_l)} \]

- In this case, we introduce a nonlinear function of the link flows into the objective function since the linear model does not take into account the economies of scale which occur as the traffic flows increase on a link.
- Note that this model would not be generally applicable to public switched telephone networks as they employ alternative routing.
• 408 paths with a maximum of 4 links per path.
• 829 decision variables
• $2.735 \times 10^{72}$ states
The Design Problem

1. Specify the network information, viz:
   - Nodes
   - Links
   - Origin - Destination pairs
   - Capacity demands for the OD pairs
   - Link costs
   - Link reliabilities

2. Generate the legal paths for the network.

3. List the OD pairs which use each link of the network as this will be required for the capacity constraint set.

4. Solve for the optimal assignment of primary and secondary paths.
• For OD pair 1 - 8, the above diagram shows the selected primary and secondary paths selected by the linear model.
• Circuits were allocated with unit module size. All demand was satisfied, even under failure conditions.
Conclusions

• The network is dimensioned on a demand basis, while minimising a cost which includes a reliability component.
• Reliability is ensured by choosing two link-disjoint paths for each OD pair in the network.
• Performance can be user specified for normal and single link failure conditions.